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THE AUTOMORPHISM GROUPS OF KUMMER SURFACES ASSOCIATED WITH THE PRODUCT OF TWO ELLIPTIC CURVES

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Abstract. We calculate the automorphism groups of several Kummer surfaces associated with the product of two elliptic curves. We give their generators explicitly.

0. Introduction

The purpose of this paper is to calculate the automorphism groups of the following Kummer surfaces X associated with the product of two elliptic curves:

Case I. $X = Km(E \times F)$, where E and F are non-isogenus generic elliptic curves.

Case II. $X = Km(E \times E)$, where E is an elliptic curve without complex multiplications.

Case III. $X = Km(E_\omega \times E_\omega)$, where ω is a 3rd root of unity and E_τ is the elliptic curve with τ as its fundamental period.

Case IV. $X = Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$.

We shall give generators of the group of automorphisms explicitly (Theorems 5.3, 5.4, 5.6, 5.7).

In the following we shall outline the proof of a theorem which is similar to that of Kondō [8] in which the second author calculated the group of automorphisms of a generic Kummer surface associated with a curve of genus two (for this case, also see Keum [5]). Recall that Kummer surfaces are $K3$ surfaces. It follows from results of Piatetski-Shapiro and Shafarevich [13] that the group of automorphisms is isomorphic, up to finite groups, to the group of isometries of the Picard lattice S_X of X which preserve the Kähler cone $D(X)$ of X. In our cases, $Aut(X)$ is infinite (Keum [6]; also see Shioda and Inose [15] for Cases III, IV) and hence it is difficult to describe the Kähler cone explicitly.

On the other hand, we can embed S_X into an even unimodular lattice $II_{1,25}$ of signature (1,25) such that the orthogonal complement R of S_X in $II_{1,25}$ is a root lattice. Conway [3] determined a fundamental domain D of the reflection group

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of $II_{1,25}$ in terms of Leech roots. Following Borcherds [1], by restricting D to the Picard lattice S_X, we can find a polyhedral cone D' in the Kähler cone $D(X)$ of X bounded by finitely many faces. Since R is a root lattice, we can easily calculate all faces of D' explicitly (Lemmas 3.2, 3.6). Next we shall carry out the most complicated part, i.e., for each face of D' we shall construct an automorphism of X which works like a reflection with respect to this face (§4). A standard argument shows that $Aut(D')$ and these automorphisms generate a subgroup of finite index in the orthogonal group $O(S_X)$.

Our method can be applied to other cases as long as the Picard lattice is the orthogonal complement of a root lattice in $II_{1,25}$.

We remark that in Case I the group generated by *all* reflections of S_X is of finite index in the orthogonal group of S_X, and in other cases it is not.

For basic definitions and results for lattices and Leech lattices, we refer the reader to the expository Sections 1 and 2 of the second author's previous paper [8].

1. Kummer surfaces

Let E and F be elliptic curves. Let p_i, q_i $(1 \leq i \leq 4)$ be 2-torsion points of E, F respectively. The quotient surface of $E \times F$ by the inversion $(-1_E, -1_F)$ has sixteen ordinary nodes corresponding to sixteen 2-torsion points (p_i, q_j) of $E \times F$. Let $X = Km(E \times F)$ be its minimal resolution, which is called the *Kummer surface* associated with $E \times F$.

It is well known that $X = Km(E \times F)$ is a $K3$ surface, i.e. it is simply connected and its canonical line bundle is trivial. There is a canonical structure of lattice on $H^2(X, \mathbb{Z})$ with the cup product which is an even unimodular lattice of signature (3, 19), and hence isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the even unimodular lattice of signature (1, 1) and E_8 the even unimodular lattice of signature (0, 8). Let S_X be the *Picard lattice* of X:

$$S_X = \{x \in H^2(X, \mathbb{Z}) : \langle x, \omega_X \rangle = 0\},$$

where ω_X is a nowhere vanishing holomorphic 2-form on X. Let T_X be the orthogonal complement of S_X, which is called the *transcendental lattice* of X. In our case

$$T_X = \begin{cases} U(2)^{\oplus 2} & \text{(Case I);} \\ U(2) \oplus \langle 4 \rangle & \text{(Case II);} \\ \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} & \text{(Case III);} \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} & \text{(Case IV).} \end{cases}$$

It follows from Nikulin [11], Theorem 3.1 that the restriction of $Aut(X)$ on T_X is a direct sum of irreducible representations of a cyclic group of order m defined over $\mathbb{Q}$ which has no nontrivial fixed vectors in $T_X \otimes \mathbb{C}$. If $\rho(X) \geq 18$, then this implies that $\varphi(m) \leq 4$, where φ is the Euler function. In Case I, we call $X = Km(E \times F)$ *generic* if $m = 2$.

Let $P(S_X)$ be the connected component of the cone $\{x \in S_X \otimes \mathbb{R} : \langle x, x \rangle > 0\}$ which contains an ample divisor. Put

$$D(X) = \{x \in P(S_X) : \langle x, \delta \rangle \geq 0 \text{ for all classes } \delta \text{ of smooth rational curves}\},$$

which is called the *Kähler cone* of X and is a fundamental domain with respect to the natural action of $W(S_X)^{(2)}$ on $P(S_X)$. It follows from the Torelli theorem for $K3$ surfaces due to Piatetski-Shapiro and Shafarevich [13] that $Aut(X)$ is isomorphic to $O(S_X)/(W(S_X)^{(2)} \cdot \{\pm 1\})$ up to finite groups. Now we have

Lemma 1.1. *In Case* I, *assume that X is generic. Let*

$$G(S_X) = \{g \in O(S_X) : g^* \mid A_{S_X} = 1\}.$$

Then $Aut(X)$ is an extension of $G(S_X)/(W(S_X)^{(2)} \cdot \{\ \pm 1\})$ by a cyclic group of order m, where $m = 2, 2, 6$ or 4 for Case I, II, III *or* IV *respectively.*

2. Picard lattice

Let E and F be elliptic curves. Let $X = Km(E \times F)$ be the Kummer surface associated with $E \times F$. Obviously X contains sixteen mutually disjoint smooth rational curves $\{G_{ij}\}$ obtained by resolution of singularities (G_{ij} corresponds to (p_i, q_j)). Let E_i (resp. F_i) be the image of $E \times \{q_i\}$ (resp. $\{p_i\} \times F$). Then $\{E_i, F_j\}$ are mutually disjoint smooth rational curves on X. The incidence relation between these curves is

$$G_{ij} \cdot E_k = \delta_{j,k}, \qquad G_{ij} \cdot F_k = \delta_{i,k}.$$

We assume that E_1, E_2, E_3 or E_4 (resp. F_1, F_2, F_3 or F_4) corresponds to $E \times \{0\}, E \times \{1/2\}, E \times \{\tau'/2\}$ or $E \times \{(1+\tau')/2\}$ respectively (resp. $\{0\} \times F, \{1/2\} \times F, \{\tau/2\} \times F$ or $\{(1+\tau)/2\} \times F$) where τ (resp. τ') is the fundamental period of E (resp. F).

In Cases II, III, IV, we denote by D_1 the nodal curve which is the image of the diagonal of $E \times E$, and by D_2, D_3 or D_4 the translation of D_1 by the 2-torsion point $(1/2, 0), (\tau/2, 0)$ or $((\tau+1)/2, 0)$ respectively. We have two families of mutually disjoint nodal curves,

$$\{G_{ij}\} \quad \text{and} \quad \{E_i, F_j, D_k\}.$$

Each member of the first family (resp. the second one) meets 3 (resp. 4) members in the second one (resp. the first one):

$$D_1 \text{ meets } G_{11}, G_{22}, G_{33}, G_{44};$$
$$D_2 \text{ meets } G_{12}, G_{21}, G_{34}, G_{43};$$
$$D_3 \text{ meets } G_{13}, G_{31}, G_{24}, G_{42};$$
$$D_4 \text{ meets } G_{14}, G_{41}, G_{23}, G_{32}.$$

In Case III, further, we denote by C_i the nodal curve which is the image of $\{p_i\} \times E_\omega$ under the map induced from the automorphism of $E_\omega \times E_\omega$ given by

$$(x, y) \longrightarrow (\omega(-x+y), -\omega^2 y).$$

We have two families consisting of sixteen mutually disjoint nodal curves,

$$\{G_{ij}\} \quad \text{and} \quad \{E_i, F_j, D_k, C_l\},$$

such that each member of one family meets exactly 4 members in the other family:

$$C_1 \text{ meets } G_{11}, G_{23}, G_{34}, G_{42};$$
$$C_2 \text{ meets } G_{14}, G_{22}, G_{31}, G_{43};$$
$$C_3 \text{ meets } G_{12}, G_{24}, G_{33}, G_{41};$$
$$C_4 \text{ meets } G_{13}, G_{21}, G_{32}, G_{44}.$$

In Case IV, there exist, further, twelve nodal curves $\{L_i\}$ such that each member of $\{G_{ij}, E_i, F_j, D_k, L_l\}$ meets six members in this family. The configuration of such nodal curves was given in Kondō [7], Example IV (see Figure 4.3. Unfortunately there is a misprint in Figure 4.3. The incident relations of E_9^+ and E_9^- (resp. E_{10}^+ and E_{10}^-) with $E_6^\pm$, $E_8^\pm$ should be changed). For example, G_{11}, G_{12}, G_{13}, G_{14}, G_{21}, G_{22}, G_{23}, G_{24}, G_{31}, G_{32}, G_{33}, G_{34}, G_{41}, G_{42}, G_{43}, G_{44} correspond to E_{11}^+, E_{12}^+, E_{10}^+, E_9^+, E_{11}^-, E_{12}^-, E_{10}^-, E_9^-, E_{14}^+, E_{13}^-, E_{15}^-, E_{16}^+, E_{13}^+, E_{14}^-, E_{16}^-, E_{15}^+ respectively; E_i $(1 \le i \le 4)$ correspond to E_1^+, E_3^+, E_2^-, E_4^- respectively; F_j $(1 \le j \le 4)$ correspond to E_{17}^+, E_{17}^-, E_{20}^+, E_{20}^- respectively; D_k $(1 \le k \le 4)$ correspond to E_5^-, E_7^+, E_8^+, E_6^- respectively; L_l $(1 \le l \le 12)$ correspond to E_2^+, E_4^+, E_{19}^-, E_{19}^+, E_6^+, E_8^-, E_5^+, E_7^-, E_{18}^-, E_{18}^+, E_3^-, E_1^- respectively. Here we assume that

$$L_1 \text{ meets } G_{13}, G_{23}, G_{34}, G_{44}, E_1, E_2;$$
$$L_2 \text{ meets } G_{14}, G_{24}, G_{33}, G_{43}, E_1, E_2;$$
$$L_3 \text{ meets } G_{31}, G_{32}, G_{43}, G_{44}, F_1, F_2;$$
$$L_4 \text{ meets } G_{33}, G_{34}, G_{41}, G_{42}, F_1, F_2;$$
$$L_5 \text{ meets } G_{13}, G_{24}, G_{32}, G_{41}, D_1, D_2;$$
$$L_6 \text{ meets } G_{14}, G_{23}, G_{31}, G_{42}, D_1, D_2;$$
$$L_7 \text{ meets } G_{12}, G_{21}, G_{33}, G_{44}, D_3, D_4;$$
$$L_8 \text{ meets } G_{11}, G_{22}, G_{34}, G_{43}, D_3, D_4;$$
$$L_9 \text{ meets } G_{11}, G_{12}, G_{23}, G_{24}, F_3, F_4;$$
$$L_{10} \text{ meets } G_{13}, G_{14}, G_{21}, G_{22}, F_3, F_4;$$
$$L_{11} \text{ meets } G_{12}, G_{22}, G_{31}, G_{41}, E_3, E_4;$$
$$L_{12} \text{ meets } G_{11}, G_{21}, G_{32}, G_{42}, E_3, E_4.$$

Lemma 2.1. *The above nodal curves generate the Picard lattice S_X.*

Proof. First consider Case I. Let $\pi : X \longrightarrow \mathbb{P}^1$ be an elliptic fibration defined by the linear system $|\ 2E_1 + G_{11} + G_{21} + G_{31} + G_{41}\ |$, which has four singular fibers of type I_0^* (in Kodaira's sense) and four sections F_1, F_2, F_3, F_4. By Shioda [14], Corollaries 1.5, 1.7, the rank of the group of sections of π is equal to 0 and its order is 4. Since S_X is generated by classes of irreducible components of fibres of π and sections, we have the assersion for Case I. The remaining cases can be proved similarly by considering the following elliptic fibrations.

Case II. Consider the elliptic fibration defined by

$$|\ G_{31} + G_{41} + G_{13} + G_{43} + 2(E_1 + G_{21} + F_2 + G_{23} + E_3)\ |,$$

which has three singular fibers of type I_4^*, I_1^*, I_0^* and two sections F_1, F_3;

Case III. Consider the elliptic fibration defined by

$$|\ G_{11} + D_1 + G_{44} + C_4 + G_{13} + E_3 + G_{23} + F_2 + G_{24} + C_3 + G_{41} + E_1\ |,$$

which has four singular fibers of type I_{12}, I_4, I_3, I_3 and six sections $G_{12}, G_{31}, G_{32}, G_{43}$ plus two hidden ones.

Case IV. Consider the elliptic fibration defined by

$$|\ E_1 + E_2 + L_1 + L_2\ |,$$

which has six singular fibers of type I_4 and sixteen sections $\{G_{ij}\}$. □

3. A relation between the Picard lattice and Leech roots

Let L be an even unimodular lattice of signature (1, 25), and Λ the Leech lattice, that is, the even unimodular lattice of signature (0, 24) without a root. In the following we fix a decomposition $L = \Lambda \oplus U$, and take $w = (0,0,1)$ as a Weyl vector. We use the same notation as in Section 2 of [8]. Consider the following vectors in Λ:

$$X = 4\nu_\infty + \nu_\Omega, \qquad Y = 4\nu_0 + \nu_\Omega, \qquad Z = 0,$$
$$P = 4\nu_\infty + 4\nu_0, \qquad Q = \nu_\Omega - 4\nu_2,$$
$$R_1 = 2\nu_{K_1}, \qquad R_2 = 2\nu_{K_2},$$
$$R_3 = (x_\infty, x_0, x_1, x_2, x_{j_1}, ..., x_{j_4}, x_{j_5}, ..., x_{j_{22}}) = (3,3,3,-1,...,-1,1,...,1)$$

where K_1, K_2 and $K_3 = \{\infty, 0, 1, 2, j_1, ..., j_4\} \in \mathcal{C}(8), K_1, K_2 \ni \infty, 0, 1, 2$.

Let

$$x = (X,1,2), \qquad y = (Y,1,2), \qquad z = (0,1,-1),$$
$$p = (P,1,1), \qquad q = (Q,1,1),$$
$$r_1 = (R_1,1,1), \qquad r_2 = (R_2,1,1), \qquad r_3 = (R_3,1,2)$$

be the corresponding Leech roots. We denote by R the sublattice of L generated by

Case I. $\{x, y, z, p, q, r_1, r_2, r_3\}$;

Case II. $\{x, y, z, p, q, r_1, r_3\}$;

Case III. $\{x, y, z, p, q, r_3\}$;

Case IV. $\{x, y, z, p, q, r_1\}$.

It is esay to see that R is isometric to $D_4 \oplus D_4$, $D_4 \oplus A_3$, $D_4 \oplus A_2$ or $A_3 \oplus A_3$ for Case I, II, III or IV respectively.

Lemma 3.1. *R is a primitive sublattice of L.*

Proof. Consider Case I. Note that $A_R \simeq (\mathbb{Z}/2\mathbb{Z})^4$ is generated by $(x-y)/2$, $(x-r_3)/2, (p-r_1)/2, (p-r_2)/2$. A direct calculation shows that $R^* \cap L = R$. This implies the assertion. The proofs of other cases are similar. □

Let S be the orthogonal complement of R in L. Since $q_R \simeq q_{T_X}$, $S \simeq S_X$ (Nikulin [12], Theorem 1.14.2). Now put

$$D' = D \cap P(S).$$

Then it is known that D' is non-empty, has only a finite number of faces, and is of finite volume. Moreover, D' contains the projection w' of the Weyl vector $w = (0,0,1) \in \Lambda \oplus U$ as an interior point (Borcherds [1], Lemmas 4.1, 4.2, 4.3). First we determine the Leech roots orthogonal to R.

Lemma 3.2. *There are exactly twenty-four, twenty-eight, thirty-two or forty Leech roots which are orthogonal to R for Case* I, II, III *or* IV *respectively.*

Proof. We consider Case I only. The other cases can be proved similarly. By Conway and Sloane [4], Table 4.13, which is reformulated in [8] as Proposition 2.3 and Remark 2.4, we can see that such a Leech root r corresponds to a vector $2\nu_K$

in Λ, where $K \in \mathcal{C}(8), 0, \infty \in K, 2 \notin K, |K \cap K_i| = 4, i = 1, 2$ and $K \cap K_3 = \{\infty, 0\}$ or $\{\infty, 0, 1, *\}$.

The number of these octads is counted as follows:

Case 1. $K \cap K_3 = \{\infty, 0\}$. Take a point A in $K_1 \setminus K_3$ and a point B in $K_2 \setminus K_3$. Then the set $\{0, \infty, A, B\}$ determines a unique sextet, or, in other words, there exist exactly five octads containing 0, ∞, A, B. Note that any two distinct octads meet in 0, 2 or 4 points because the $\mathcal{C}$-set consists of 0, 8, 12, 16 or 24 elements. It follows easily from this fact that the above five octads satisfy the condition $|K \cap K_i| = 4$ for $i = 1, 2$ and exactly two of them satisfy the condition $K \cap K_3 = \{\infty, 0\}$. Let K be a desired octad. Let A' be in Ω with $K \cap K_1 = \{\infty, 0, A, A'\}$. Then K also appears when we consider the sextet defined by $\{\infty, 0, A', B\}$. Hence the number of desired octads is $(\binom{4}{1} \times \binom{4}{1} \times 2)/(2 \times 2) = 8$.

Case 2. $K \cap K_3 = \{\infty, 0, 1, *\}$. Take a point A in $K_3 \setminus K_1$. Then $\{\infty, 0, 1, A\}$ determines exactly five octads containing these four points. It is now easy to see that these five octads satisfy the condition $|K \cap K_i| = 4$. Here we need to exclude K_3. Hence the desired number of octads is eqaul to $\binom{4}{1} \times (5 - 1) = 16$. □

Remark 3.3. In his paper [16], Todd gave the 759 octads of the Steiner system $S(5, 8, 24)$ (p. 219, Table I). The octads in the proof of Lemma 3.2 correspond to the following $\{E_i, F_i, G_{ij}\}$. Here we use the same symbol as nodal curves stated in §2 because these can be identified (see §5). We assume

$$K_1 = \{\infty, 0, 1, 2, 3, 5, 14, 17\},$$
$$K_2 = \{\infty, 0, 1, 2, 4, 13, 16, 22\},$$
$$K_3 = \{\infty, 0, 1, 2, 6, 7, 19, 21\}.$$

Then

Case I.

$$G_{11} = \{\infty, 0, 1, 3, 4, 11, 19, 20\}, \quad G_{21} = \{\infty, 0, 1, 3, 6, 8, 10, 13\},$$
$$G_{12} = \{\infty, 0, 1, 3, 7, 9, 16, 18\}, \quad G_{22} = \{\infty, 0, 1, 3, 12, 15, 21, 22\},$$
$$G_{31} = \{\infty, 0, 1, 4, 5, 7, 8, 15\}, \quad G_{14} = \{\infty, 0, 1, 4, 6, 9, 12, 17\},$$
$$G_{34} = \{\infty, 0, 1, 4, 10, 14, 18, 21\}, \quad G_{42} = \{\infty, 0, 1, 5, 6, 18, 20, 22\},$$
$$G_{41} = \{\infty, 0, 1, 5, 9, 11, 13, 21\}, \quad G_{32} = \{\infty, 0, 1, 5, 10, 12, 16, 19\},$$
$$G_{33} = \{\infty, 0, 1, 6, 11, 14, 15, 16\}, \quad G_{23} = \{\infty, 0, 1, 7, 10, 11, 17, 22\},$$
$$G_{44} = \{\infty, 0, 1, 7, 12, 13, 14, 20\}, \quad G_{43} = \{\infty, 0, 1, 8, 9, 14, 19, 22\},$$
$$G_{13} = \{\infty, 0, 1, 8, 16, 17, 20, 21\}, \quad G_{24} = \{\infty, 0, 1, 13, 15, 17, 18, 19\},$$
$$E_3 = \{\infty, 0, 3, 4, 5, 12, 13, 18\}, \quad F_4 = \{\infty, 0, 3, 4, 10, 15, 16, 17\},$$
$$E_4 = \{\infty, 0, 3, 5, 8, 11, 16, 22\}, \quad F_3 = \{\infty, 0, 3, 9, 13, 17, 20, 22\},$$
$$F_2 = \{\infty, 0, 4, 5, 9, 14, 16, 20\}, \quad E_2 = \{\infty, 0, 4, 8, 11, 13, 14, 17\},$$
$$F_1 = \{\infty, 0, 5, 10, 13, 14, 15, 22\}, \quad E_1 = \{\infty, 0, 12, 14, 16, 17, 18, 22\};$$

Case II. G_{ij}, E_i, F_j and

$$D_4 = \{\infty, 0, 3, 8, 14, 15, 18, 20\}, \quad D_3 = \{\infty, 0, 3, 9, 10, 11, 12, 14\},$$
$$D_1 = \{\infty, 0, 5, 8, 9, 10, 17, 18\}, \quad D_2 = \{\infty, 0, 5, 11, 12, 15, 17, 20\};$$

Case III. G_{ij}, E_i, F_j, D_k and

$$C_3 = \{\infty, 0, 4, 8, 10, 12, 20, 22\}, \quad C_4 = \{\infty, 0, 4, 9, 11, 15, 18, 22\},$$
$$C_1 = \{\infty, 0, 8, 9, 12, 13, 15, 16\}, \quad C_2 = \{\infty, 0, 10, 11, 13, 16, 18, 20\};$$

Case IV. G_{ij}, E_i, F_j, D_k and

$$L_1 = \{\infty, 0, 3, 5, 6, 9, 15, 19\}, \quad L_2 = \{\infty, 0, 3, 5, 7, 10, 20, 21\},$$
$$L_3 = \{\infty, 0, 3, 6, 11, 17, 18, 21\}, \quad L_4 = \{\infty, 0, 3, 7, 8, 12, 17, 19\},$$
$$L_5 = \{\infty, 0, 3, 4, 6, 7, 14, 22\}, \quad L_6 = \{\infty, 0, 3, 13, 14, 16, 19, 21\},$$
$$L_7 = \{\infty, 0, 4, 5, 17, 19, 21, 22\}, \quad L_8 = \{\infty, 0, 5, 6, 7, 13, 16, 17\},$$
$$L_9 = \{\infty, 0, 5, 6, 8, 12, 14, 21\}, \quad L_{10} = \{\infty, 0, 5, 7, 11, 14, 18, 19\},$$
$$L_{11} = \{\infty, 0, 6, 10, 14, 17, 19, 20\}, \quad L_{12} = \{\infty, 0, 7, 9, 14, 15, 17, 21\}.$$

For simplicity we use the same symbols $E_i, F_j, G_{ij}, C_k, L_l$ for the corresponding Leech roots.

Lemma 3.4. *Let w' be the projection of the Weyl vector $w = (0, 0, 1) \in \Lambda \oplus U$. Then*

Case I. $w' = (\sum 3(E_i + F_i) + \sum 2G_{ij})/2$, $(w')^2 = 28$,

Case II. $w' = (\sum 6(E_i + F_i + D_i) + \sum 5G_{ij}))/8$, $(w')^2 = 19$,

Case III. $w' = \sum(E_i + F_i + D_i + C_i + G_{ij})/2$, $(w')^2 = 16$,

Case IV. $w' = \sum(E_i + F_i + D_i + G_{ij} + L_k)/4$, $(w')^2 = 10$.

Proof. We shall give a proof in Case I. The other cases are similar. It is easy to see that both sides of the equation in Lemma 3.4 have the self-intersection number 28 and meet all E_i, F_j, G_{ij} with multiplicity one. Since $\{E_i, F_j, G_{ij}\}$ generate S (Lemma 2.1), this implies the assertion. □

We denote by $Aut(D')$ the group of isometries of S which preserves D'. Let $Sym(R)$ be the group of symmetries of the Dynkin diagram of R. Obviously the canonical map $Sym(R) \longrightarrow O(q_R)$ is surjective. It follows from Nikulin [12], Proposition 1.6.1, that each isometry $\varphi \in Aut(D')$ can be extended to an isometry of L which preserves D. Hence $Aut(D')$ is isomorphic to the subgroup G of $Aut(D)$ which preserves the Dynkin diagram of R. We denote by G_0 the subgroup of G which acts trivially on R. We remark that if we identify the set of nodal curves given in §2 with the set of Leech roots in Lemma 3.2, then G_0 can be considered as a group of symplectic automorphisms of X, i.e. they act on T_X trivially.

Lemma 3.5.

Case I. $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^4$, $G/G_0 \simeq O^+(4, \mathbb{F}_2) \simeq Sym(R)$.

Case II. $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathbb{Z}/3\mathbb{Z}$, $G/G_0 \simeq S_3 \times \mathbb{Z}/2\mathbb{Z} \simeq Sym(R)$.

Case III. $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot A_4$, $G/G_0 \simeq S_3 \times \mathbb{Z}/2\mathbb{Z} \simeq Sym(R)$.

Case IV. $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot A_5$, $G/G_0 \simeq D_8$, *dihedral of order* 8, $\simeq Sym(R)$.

Proof. First consider Case I. Obviously the restriction of G on R is a subgroup of the symmetry group $(S_3 \times S_3) \cdot \mathbb{Z}/2\mathbb{Z} \simeq O^+(4, \mathbb{F}_2)$ of the Dynkin diagram of R. G_0 is a subgroup of M_{22} because G_0 fixes X, Y, Z and P (see Conway and Sloane [4], Chap.10, §3.5). Moreover, G_0 preserves three octads K_i and fixes 1, 2. By the table of maximal subgroups of M_{22} (Conway [2], Table 3), G_0 is isomorphic to a 2-elementary abelian group of order 16. Thus G is a subgroup of the group $(\mathbb{Z}/2\mathbb{Z})^4 \cdot ((S_3 \times S_3) \cdot \mathbb{Z}/2\mathbb{Z})$. On the other hand, $O(S) \simeq O(S_X)$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \cdot ((S_3 \times S_3) \cdot \mathbb{Z}/2\mathbb{Z})$ induced from the automorphism group $Aut(\Gamma)$ of the graph Γ whose vertices are $\{E_i, F_j, G_{ij}\}$, where $(\mathbb{Z}/2\mathbb{Z})^4$ is the group of translations by 2-torsion points, $S_3 \times S_3$ is generated by permutations on $\{E_1, E_2, E_3\}$ or $\{F_1, F_2, F_3\}$, and $\mathbb{Z}/2\mathbb{Z}$ is generated by an involution interchanging E_i and F_i. By Lemma 3.4, they preserve w' and hence D'. Thus $Aut(D') \simeq G \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot ((S_3 \times S_3) \cdot \mathbb{Z}/2\mathbb{Z})$.

In Cases II, III, IV, the proof is similar. Here we only give generators of G_0 and G/G_0, and leave the details to the reader.

Case II. G_0 is generated by translations by 2-torsions and an automorphism τ of order 3 induced from the automorphism of $E \times E$ given by

$$(x, y) \longrightarrow (x - y, x).$$

G/G_0 is generated by an automorphism σ which is induced from the automorphism of $E \times E$ given by

$$(x, y) \longrightarrow (y, x)$$

and the permutations on $\{E_i, F_j, D_k\}$ given by

$$E_1 \leftrightarrow E_2, \qquad F_1 \leftrightarrow F_2, \qquad D_3 \leftrightarrow D_4,$$

and

$$E_1 \leftrightarrow E_3, \qquad F_1 \leftrightarrow F_3, \qquad D_2 \leftrightarrow D_4.$$

Case III. G_0 is generated by translations by 2-torsions and automorphisms induced from

$$(x, y) \longrightarrow (y, -x + y)$$

and

$$(x, y) \longrightarrow (\omega(-x + y), -\omega^2 y).$$

We remark that A_4 is isomorphic to the alternating group on the set of four families $\{E_i\}, \{F_j\}, \{D_k\}, \{C_l\}$. S_3 is generated by an automorphism induced from that of $E_\omega \times E_\omega$ given by

$$(x, y) \longrightarrow (\omega x, \omega y)$$

and a transposition on the above set of four families, given by

$$E_2 \leftrightarrow E_3, \qquad F_3 \leftrightarrow F_4.$$

$\mathbb{Z}/2\mathbb{Z}$ in G/G_0 is induced from an isometry σ which interchanges $\{G_{ij}\}$ and $\{E_i$, F_j, D_k, $C_l\}$. Later we shall see that σ is induced from an automorphism of X.

Case IV. From the configuration of forty nodal curves on X one gets exactly five elliptic fibrations which has six singular fibers of type I_4 and sixteen sections, namely

$$| E_1 + E_2 + L_1 + L_2 |, \qquad | G_{11} + G_{12} + F_1 + L_9 |, \qquad | G_{11} + G_{21} + E_1 + L_{12} |,$$
$$| G_{11} + G_{22} + D_1 + L_8 |, \qquad | G_{33} + G_{44} + D_1 + L_7 | .$$

The G_0 is generated by automorphisms induced from translations by sections of these elliptic fibrations. On the other hand, there exists a Leech root r such that the Dynkin diagram of $\{x, y, z, p, q, r_1, r\}$ is isomorphic to A_7 or D_7 (see the following Lemma 3.6). By Borcherds [1], Lemma 9.6, Theorem 9.5, there is an isometry in $Aut(D)$ whose restriction on R is a symmetry of the Dynkin diagram of R. These two isometries generate $G/G_0 \simeq D_8$. □

Next we shall classify the hyperplanes bounding D'. It suffices to consider Leech roots r such that r and R generate a root lattice R' with $rank(R') = rank(R) + 1$, because otherwise the hyperplane orthogonal to r does not meet $P(S)$. Obviously R' is isomorphic to

Case I. $D_4 \oplus D_4 \oplus A_1$ or $D_4 \oplus D_5$;

Case II. $D_4 \oplus A_3 \oplus A_1$, $D_4 \oplus D_4$, $D_4 \oplus A_4$, $D_5 \oplus A_3$ or D_8;

Case III. $D_4 \oplus A_2 \oplus A_1$, $D_4 \oplus A_3$, $D_5 \oplus A_2$ or D_7;

Case IV. $A_3 \oplus A_3 \oplus A_1$, $D_4 \oplus A_3$, $A_4 \oplus A_3$, A_7 or D_7.

We have already taken care of the first case $R' = R \oplus A_1$ in Lemma 3.2.

Lemma 3.6. *Let r be a Leech root. We denote by r' the projection of r into S^*. Then the number of Leech roots and $\langle r', r'\rangle$ is as follows:*

Case I-(1). $(R' = D_4 \oplus D_5) : 24$, $\langle r', r'\rangle = -1$.

Case II-(1). $(R' = D_4 \oplus D_4) : 3$, $\langle r', r'\rangle = -1$.
(2) $(R' = D_4 \oplus A_4) : 8$, $\langle r', r'\rangle = -5/4$.
(3) $(R' = D_5 \oplus A_3) : 36$, $\langle r', r'\rangle = -1$.
(4) $(R' = D_8) : 72$, $\langle r', r'\rangle = -1/4$.

Case III-(1). $(R' = D_4 \oplus A_3) : 8$, $\langle r', r'\rangle = -4/3$.
(2) $(R' = D_5 \oplus A_2) : 72$, $\langle r', r'\rangle = -1$.
(3) $(R' = D_7) : 96$, $\langle r', r'\rangle = -1/3$.

Case IV-(1). $(R' = D_4 \oplus A_3) : 40$, $\langle r', r'\rangle = -1$.
(2) $(R' = A_4 \oplus A_3) : 64$, $\langle r', r'\rangle = -5/4$.
(3) $(R' = A_7) : 160$, $\langle r', r'\rangle = -1/2$.
(4) $(R' = D_7) : 320$, $\langle r', r'\rangle = -1/4$.

Proof. First we shall give a proof in Case I. First assume that $\langle r, p\rangle = 1$. Then by [8], Proposition 2.3 and Remark 2.4, the root r corresponds to a vector

$$Q' = \nu_\Omega - 4\nu_k$$

in Λ, where $k \notin K_1, K_2, k \in K_3$. Hence the number of such Leech roots is equal to 4. Next, for a Leech root r with $\langle r, p\rangle = 1$, write $r = r' + r''$, where $r' \in S^*, r'' \in R^*$. Then $r'' = p^* = (-2p - 2q - r_1 - r_2)/2$. Hence $\langle r', r'\rangle = -1$. Finally, note that $Aut(D')$ acts on the set of vectors $\{x, y, p, r_1, r_2, r_3\}$ transitively (Lemma 3.5). Therefore the total number of Leech roots as above is equal to $4 \times 6 = 24$.

In the remaining cases the proof is similar. We shall give one such Leech root and leave the details to the reader. Here we use the same identification as in Remark 3.3.

Case I. r corresponds to $\nu_\Omega - 4\nu_{19}$. r meets exactly four Leech roots G_{ij}, $ij = 11, 24, 32, 43$. The projection r' is equal to

$$\left(\sum(E_i + F_i) + \sum G_{ij} - 2(G_{11} + G_{24} + G_{32} + G_{43})\right)/4.$$

Case II-(1). r corresponds to $2\nu_K$, $K = \{\infty, 0, 1, 2, 4, 13, 16, 22\}$. r meets exactly four Leech roots D_i, $1 \le i \le 4$. The projection r' is equal to

$$\left(2\sum(E_i + F_i + D_i) + \sum G_{ij} - 4\sum D_i\right)/8.$$

II-(2): The same Leech root as in Case I. r meets exactly four Leech roots G_{ij}, $ij = 11, 24, 32, 43$. The projection r' is equal to

$$(2\sum(E_i + F_i + D_i) + 3\sum G_{ij} - 8(G_{11} + G_{24} + G_{32} + G_{43}))/16.$$

II-(3): r corresponds to

$$(x_\infty, x_2, x_4, x_6, x_7, x_8, x_{12}, x_{13}, ...) = (3, -1, -1, -1, -1, -1, -1, -1, 1, ..., 1),$$

where $K = \{\infty, 2, 4, 6, 7, 8, 12, 13\}$ is an octad. r meets exactly six Leech roots G_{ij}, E_k, $ij = 14, 21, 31, 44$, $k = 2, 3$. The projection of r' is equal to

$$(\sum(F_i + D_i) + \sum G_{ij} - 2(G_{14} + G_{21} + G_{31} + G_{44}))/4$$

II-(4): r corresponds to $2\nu_K$, $K = \{\infty, 1, 3, 4, 5, 9, 10, 22\}$. r meets exactly eight Leech roots G_{ij}, E_k, D_l, $ij = 13, 24, 33, 44$, $k = 1, 2$, $l = 2, 4$. The projection r' is equal to

$$(6\sum(E_i + F_i + D_i) + 5\sum G_{ij} - 8(E_1 + E_2 + D_2 + D_4) - 4(G_{12} + G_{13} + G_{21} + G_{24} + G_{32} + G_{33} + G_{41} + G_{44}))/16.$$

Case III-(1). The same Leech root as in Case I. r meets exactly four Leech roots G_{ij}, $ij = 11, 24, 32, 43$. The projection r' is equal to

$$(\sum(E_i + F_i + D_i + C_i) + 2\sum G_{ij} - 6(G_{11} + G_{24} + G_{32} + G_{43}))/12.$$

III-(2): The same Leech root as in Case II-(3). r meets exactly eight Leech roots G_{ij}, E_k, C_l, $ij = 14, 21, 31, 44$, $k = 2, 3$, $l = 1, 3$. The projection r' is equal to

$$(\sum(F_i + D_i) + \sum G_{ij} - 2(G_{14} + G_{21} + G_{31} + G_{44}))/4.$$

III-(3): The same Leech root as in Case II-(4). r meets exactly ten Leech roots G_{ij}, E_k, D_l, C_m, $ij = 13, 24, 33, 44$, $k = 1, 2$, $l = 2, 4$, $m = 1, 2$. The projection r' is equal to

$$(2\sum(E_i + F_i + D_i + C_i) + \sum G_{ij} - 3(E_1 + E_2 + D_2 + D_4 + C_1 + C_2))/6.$$

Case IV-(1). The same Leech root as in Case II-(1). r meets exactly twelve Leech roots D_i, $1 \leq i \leq 4, L_j, j = 1, 2, 3, 4, 9, 10, 11, 12$. The projection r' is equal to

$$(2\sum(E_i + F_i + D_i) + \sum G_{ij} - 4\sum D_i)/8.$$

IV-(2): The same Leech root as in Case I. r meets exactly ten Leech roots G_{ij}, L_k, $ij = 11, 24, 32, 43$, $k = 1, 4, 6, 7, 10, 11$. The projection r' is equal to

$$(2\sum(E_i + F_i + D_i) + 4\sum L_j + \sum G_{ij} - 8(L_1 + L_4 + L_6 + L_7 + L_{10} + L_{11}))/16.$$

IV-(3): The same Leech root as in Case II-(4). r meets exactly sixteen Leech roots G_{ij}, E_k, D_l, L_m, $ij = 13, 24, 33, 44$, $k = 1, 2$, $l = 2, 4$, $m = 3, 4, 6, 8$, 9, 10, 11, 12. The projection r' is equal to

$$(2\sum G_{ij} + \sum(E_i + F_i + D_i) - \sum L_j - 2(G_{11} + G_{22} + G_{31} + G_{42} + G_{13} + G_{24} + G_{33} + G_{44}) + 4(L_1 + L_2 + L_5 + L_7))/8.$$

IV-(4): r corresponds to

$$(x_\infty, x_1, x_4, x_6, x_7, x_{10}, x_{16}, x_{20}, ...) = (3, -1, -1, -1, -1, -1, -1, -1, 1, ..., 1),$$

where $K = \{\infty, 1, 4, 6, 7, 10, 16, 20\}$ is an octad. The Leech root r meets exactly eighteen Leech roots G_{ij}, F_k, L_l, $ij = 11, 12, 13, 14, 21, 23, 31, 32, 33, 34, 42, 44$, $k = 2, 4$, $l = 2, 5, 8, 11$. The projection r' is equal to

$$(5\sum G_{ij} + 20\sum F_i + 10\sum L_i - 16(G_{22} + G_{24} + G_{41} + G_{43}) \\ -24(F_2 + F_4 + L_2 + L_5 + L_8 + L_{11}))/16.$$

□

4. Examples of automorphisms of X

In this section, we shall give examples of automorphisms of X corresponding to the Leech roots listed in Lemma 3.6 (see also the proof of Lemma 3.6).

4.1. *Case* I. Put

$$H = (\sum_i (E_i + F_i) + \sum_{i,j} G_{ij})/2.$$

Then $H^2 = 4$, $H \cdot E_i = H \cdot F_j = 1$ and $H \cdot G_{ij} = 0$ for any i, j. Recall that $2E_i + \sum_j G_{ji}$ (resp. $2F_j + \sum_i G_{ji}$) is a singular fiber of an elliptic fibration (see the proof of Lemma 2.1). From this fact we can easily see that $H \in S_X$. Put

$$r' = (H - G_{i_1j_1} - G_{i_2j_2} - G_{i_3j_3} - G_{i_4j_4})/2,$$

where $\{i_1, i_2, i_3, i_4\} = \{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$. Obviously $\langle r', r'\rangle = -1$, and the number of vectors r' of this form is 24. These r' correspond to the Leech roots in Lemma 3.6. For each r' we shall construct an involution $f_{r'}$ which sends r' to $-r'$. For simplicity we may assume that $r' = (H - G_{12} - G_{21} - G_{34} - G_{43})/2$.

First consider the elliptic pencil $\pi : X \longrightarrow \mathbb{P}^1$ defined by the linear system

$$\mid G_{13} + G_{14} + G_{31} + G_{41} + 2(G_{11} + E_1 + F_1) \mid .$$

Then π has two singular fibers of type I_2^* in Kodaira's sense, and G_{33}, G_{34}, G_{43}, G_{44} are components of other singular fibers. Since E_3, E_4, F_3, F_4 are sections of π, other singular fibers are all of type I_2 or III. It follows from Shioda [14], Corollaries 1.5, 1.7, that the Mordell-Weil group of π is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Hence the translation sending E_4 to F_4 acts on X as an involution $f_{r'}$. The action of $f_{r'}$ on twenty-four nodal curves is as follows:

$$f_{r'}(E_i) = F_i,$$

$$f_{r'}(G_{ij}) = \begin{cases} G_{ii}, & \text{if } i = j; \\ G_{ji}, & \text{if } (i,j) \notin \{(1,2),(2,1),(3,4),(4,3)\}; \\ G_{ji} + 2r', & \text{if } (i,j) \in \{(1,2),(2,1),(3,4),(4,3)\}. \end{cases}$$

It is easy to see that $f_{r'}(H) = H + 4r'$ and $f_{r'}(r') = -r'$.

Remark 4.1. Take an involution t induced from the translation of $E \times F$ by 2-torsion with $t(E_1) = E_2$, $t(F_1) = F_2$. Let ι be the involution of X induced from an automorphism of $E \times F$ given by $(x, y) \longrightarrow (x, -y)$. We shall see that $t \circ f_{r'}$ comes from a Cremona transformation of $\mathbb{P}^2$. Recall that the linear system $|H|$ gives a double cover $\pi : X \longrightarrow Q = \mathbb{P}^1 \times \mathbb{P}^1$ which is the composite of the quotient map by ι and contractions of sixteen nodal curves G_{ij}. Put $w_{ij} = \pi(G_{ij})$. We use the same symbols E_i, F_j for the images of E_i, F_j in Q. Then w_{ij} is the intersection point of E_j and F_i. First blow up at w_{21} and then contract the proper transform of

E_1 and F_2. Then we have seven lines $\{\bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{F}_1, \bar{F}_3, \bar{F}_4, L\}$ in $\mathbb{P}^2$, where $\bar{E}_i, \bar{F}_j$ are the image of the proper transforms of E_i, F_j respectively and L is the image of the exceptional divisor appearing at the first blow-up. Three lines $\bar{E}_2, \bar{E}_3, \bar{E}_4$ meet at one point (= the image of F_2), $\bar{F}_1, \bar{F}_3, \bar{F}_4$ meet at one point (= the image of E_1), and L passes through these two points. Now consider the standard Cremona transformation of $\mathbb{P}^2$ with base w_{12}, w_{34}, w_{43} which induces a birational involution $c_{r'}$ of Q. It is easy to see that $c_{r'}$ preserves the divisor $\sum_i (E_i + F_i)$, and hence it can be lifted to an involution $\tau_{r'}$ of X. We can easily see that the actions of $\tau_{r'}$ and $t \circ f_{r'}$ on S_X coincide, and hence $\tau_{r'} \equiv t \circ f_{r'} \bmod \iota$ by the Torelli theorem for $K3$ surfaces (Piatetski-Shapiro and Shafarevich [13]).

Remark 4.2. The reflection associated with the (-1)-vector r' is not represented by an automorphism of X because it does not act on A_{S_X} as ± 1.

Remark 4.3. The description of X as a double plane branched along the union of two sets of three lines meeting at one point shows that X is a degeneration of a Jacobian Kummer surface which is a Kummer surface associated with a curve of genus two. There are sixteen projections of a Jacobian Kummer surface onto $\mathbb{P}^2$, resulting in sixteen involutions called projections [5]. These sixteen involutions on a Jacobian Kummer surface become the same involution ι on X.

4.2. *Case* II-(1). We assume that

$$r' = (2\sum(E_i + F_i + D_i) + \sum G_{ij} - 4\sum D_i)/8.$$

Put

$$H = (\sum 2(E_i + F_i + D_i) + \sum G_{ij})/4.$$

It is easy to see that $H^2 = 4, H \cdot G_{ij} = 1, H \cdot E_i = H \cdot F_i = H \cdot D_i = 0$ and H is in S_X. Let $f_{r'}$ be an involution of X induced from the involution $(1_E, -1_E)$ of $E \times E$. Then $f_{r'}$ preserves each curve in $\{G_{ij}, E_i, F_j\}$ and

$$f_{r'}(D_i) = H - \sum_{j \neq i} D_j.$$

Thus $f_{r'}$ acts on S_X as a reflection with respect to the (-4)-vector $2r' = H - \sum_i D_i$.

Case II-(2). We assume that

$$r' = (2\sum(E_i + F_i + D_i) + 3\sum G_{ij} - 8(G_{11} + G_{24} + G_{32} + G_{43}))/16.$$

We shall find an automorphism $f_{r'}$ of X such that $f_{r'}(r') = -s'$ where

$$s' = (2\sum(E_i + F_i + D_i) + 3\sum G_{ij} - 8(G_{11} + G_{23} + G_{34} + G_{42}))/16$$

is the projection of a Leech root s of the same type.

Consider the elliptic pencil π on X defined by the linear system

$$\begin{aligned} |\, G_{21} + F_2 + G_{22} + E_2 + G_{12} + D_2 \,| &= |\, G_{31} + F_3 + G_{33} + E_3 + G_{13} + D_3 \,| \\ &= |\, G_{41} + F_4 + G_{44} + E_4 + G_{14} + D_4 \,| . \end{aligned}$$

The elliptic pencil π has three singular fibers of type I_6 and a singular fiber of type I_2 containg G_{11}, and has nine sections $E_1, F_1, D_1, G_{23}, G_{24}, G_{32}, G_{34}, G_{42}, G_{43}$. We remark that its Mordell-Weil group has rank 1. We consider E_1 as 0-section of π. Then the translation by G_{43} induces an automorphism $f_{r'}$ of X of infinite order. Since $E_1 \cdot G_{43} = 0$, $G_{43} \cdot f_{r'}(G_{43}) = 0$. Also $E_1 \cdot f_{r'}(G_{43}) = 0$ because $f_{r'}$

has infinite order, and hence $f_{r'}^2$ acts on smooth locus of each fiber freely. This determines $f_{r'}(G_{43})$, i.e.

$$\begin{aligned}f_{r'}(G_{43}) = G_{11} - G_{23} + (G_{13} + G_{14} + G_{21} + G_{22} + G_{33} + G_{41} - G_{34} - G_{42})/2\\ + E_1 + F_1 + D_1.\end{aligned}$$

Similary we have

$$\begin{aligned}f_{r'}(F_1) = G_{32},\ f_{r'}(D_1) = G_{24},\ f_{r'}(G_{42}) = D_1,\ f_{r'}(G_{23}) = E_1,\ f_{r'}(G_{34}) = F_1,\\ f_{r'}(G_{32}) = G_{11} - G_{34} + (G_{12} + G_{14} + G_{21} + G_{31} + G_{33} + G_{44} - G_{23} - G_{42})/2\\ + E_1 + F_1 + D_1,\\ f_{r'}(G_{24}) = G_{11} - G_{42} + (G_{12} + G_{13} + G_{22} + G_{31} + G_{41} + G_{44} - G_{23} - G_{34})/2\\ + E_1 + F_1 + D_1.\end{aligned}$$

A direct calculation shows that $f_{r'}(r') = -s'$. We remark that $f_{s'}$ is given by the translation by G_{23}, and $f_{r'}^{-1} = f_{s'}$.

Case II-(3). We assume that

$$r' = (\sum(F_i + D_i) + \sum G_{ij} - 2(G_{14} + G_{21} + G_{31} + G_{44}))/4.$$

By interchanging E_i and D_i, we have the same situation as in Case I. Let π be the elliptic pencil on X defined by the linear system

$$\begin{aligned}&|\ G_{12} + G_{13} + G_{22} + G_{33} + 2(F_1 + D_1 + G_{11})\ |\\ &= |\ G_{23} + G_{32} + G_{42} + G_{43} + 2(F_4 + D_4 + G_{41})\ | = |\ G_{21} + G_{21}^{'}\ |\\ &= |\ G_{31} + G_{31}^{'}\ | = |\ G_{24} + G_{34} + E_4 + E_4^{'}\ |,\end{aligned}$$

where $G_{21}^{'}, G_{31}^{'}, E_4^{'}$ are nodal curves. π has two singular fibers of type I_2^*, two of type I_2 and one of type I_4, and its Mordell-Weil group is $(\mathbb{Z}/2\mathbb{Z})^2$. Let $f_{r'}$ be an involution induced from the translation by sending F_2 to F_3. We can easily see that

$$G_{21}^{'} = G_{31} + 2r', \quad G_{31}' = G_{21} + 2r', \quad G_{14}' = G_{14} + 2r', \quad G_{44}' = G_{44} + 2r'.$$

Thus we have $f_{r'}(r') = -r'$.

Case II-(4). We assume that

$$\begin{aligned}r' = (6\sum(E_i + F_i + D_i) + 5\sum G_{ij} - 8(E_1 + E_2 + D_2 + D_4)\\ - 4(G_{12} + G_{13} + G_{21} + G_{24} + G_{32} + G_{33} + G_{41} + G_{44}))/16.\end{aligned}$$

Consider the following twenty nodal curves which are orthogonal to r': G_{ij}, E_k, F_l, D_m, ij = 11, 12, 14, 21, 22, 23, 31, 32, 34, 41, 42, 43, $k = 3, 4$, $l = 1, 2, 3, 4$, $m = 1, 3$. The dual graph of these curves except $G_{12}, G_{21}, G_{32}, G_{41}$ is the same as that of singular fiber of an elliptic pencil π of type I_{16}, and $G_{12}, G_{21}, G_{32}, G_{41}$ are sections of π. This diagram appears in the classification of cohomologically trivial automorphisms of Enriques surfaces (Mukai and Namikawa [10], 4.2; also see Kondō [7], 1.7). The linear system

$$\begin{aligned}|\ H\ | =|\ G_{21} + G_{23} + G_{31} + G_{32} + 2(G_{12} + G_{22} + G_{34} + E_4 + F_2 + F_3 + D_1)\\ + 3(G_{11} + G_{14}) + 4F_1\ |\end{aligned}$$

gives a morphism of degree 2 from X to a smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$ (E_3, E_4, D_1, D_3 are rulings). Denote by $f_{r'}$ its covering transformation. Since $X/\langle f_{r'} \rangle$ is rational,

$f_{r'}$ acts on $H^0(X, \Omega_X^2)$ as -1. This implies that the fixed locus of $f_{r'}$ is a disjoint union of smooth curves. Obviously $f_{r'}$ fixes $F_1, F_2, F_3, F_4, E_3, E_4, D_1, D_3$, and acts on each G_{ij}, $ij = 11, 12, 14, 21, 22, 23, 31, 32, 34, 41, 42, 43$, as an involution. Thus $f_{r'}$ acts trivially on the sublattice of rank 18 generated by these nodal curves. Since $f_{r'}$ acts on a section as involution, there is a unique other fiber F of π invariant under $f_{r'}$. Therefore $f_{r'}$ fixes a component C of F which meets G_{ij}, $ij = 12, 21, 32$ or 41. We shall see that C is a smooth elliptic curve. Assume that F is reducible. Then it is of type I_2 or III because the Picard number of X is 19. Thus $C \cdot G_{ij} = 1$ for all $ij = 12, 21, 32, 41$, and hence the image of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is a smooth divisor of bidegree (2,2). This is a contradiction. Hence $C = F$ is smooth. Thus we have the same configuration of fixed curves as in Mukai and Namikawa [10], (4.2). Since $f_{r'}$ acts on the orthogonal complement of this sublattice as -1 (Mukai and Namikawa [10], Proposition 2.5), $f_{r'}$ acts on S_X as a reflection with respect to r'.

4.3. *Case* III-(1). We assume that

$$r' = (\sum(E_i + F_i + D_i + C_i) + 2\sum G_{ij} - 6(G_{11} + G_{24} + G_{32} + G_{43}))/12.$$

This case is similar to Case II-(2). We shall define an automorphism $f_{r'}$ such that $f_{r'}(r') = -s'$, where

$$s' = (\sum 2(E_i + F_i + D_i + C_i) + \sum G_{ij} - 6(C_1 + C_2 + C_3 + C_4))/12$$

is the projection of a Leech root s of the same type. Consider the elliptic pencil defined by the same linear system as in Case II-(2). This elliptic pencil has three singular fibers of type I_6 and one singular fiber of type I_3 containing G_{11}, C_1, and twelve sections. The translation by sending E_1 to G_{43} induces an automorphism $f_{r'}$:

$$\begin{aligned}
&f_{r'}(C_2) = G_{23}, \qquad f_{r'}(C_3) = G_{42}, \qquad f_{r'}(C_4) = G_{34},\\
&f_{r'}(G_{43}) = G_{11} - G_{23} + (G_{13} + G_{14} + G_{21} + G_{22} + G_{33} + G_{41} - G_{34} - G_{42})/2\\
&\qquad\qquad + E_1 + F_1 + D_1 - C_1,\\
&f_{r'}(G_{32}) = G_{11} - G_{34} + (G_{12} + G_{14} + G_{21} + G_{31} + G_{33} + G_{44} - G_{23} - G_{42})/2\\
&\qquad\qquad + E_1 + F_1 + D_1 - C_1,\\
&f_{r'}(G_{24}) = G_{11} - G_{42} + (G_{12} + G_{13} + G_{22} + G_{31} + G_{41} + G_{44} - G_{23} - G_{34})/2\\
&\qquad\qquad + E_1 + F_1 + D_1 - C_1.
\end{aligned}$$

A direct calculation shows that $f_{r'}(r') = -s'$.

Case III-(2). We assume that

$$r' = (\sum(F_i + D_i) + \sum G_{ij} - 2(G_{14} + G_{21} + G_{31} + G_{44}))/4.$$

This case is similar to Case II-(3). The same translation induces an automorphism $f_{r'}$ whose action on $\{G_{ij}, E_i, F_j, D_k\}$ coincides with that of Case II-(3). Since r' depends only on $\{G_{ij}, E_i, F_j, D_k\}$, $f_{r'}$ sends r' to $-r'$.

Case III-(3). We assume that

$$r' = (2\sum(E_i + F_i + D_i + C_i) + \sum G_{ij} - 3(E_1 + E_2 + D_2 + D_4 + C_1 + C_2))/6.$$

This case is similar to Case II-(4). We use the same notation as in Case II-(4). Consider the elliptic pencil π defined by the same linear system as in Case II-(4).

π has a singular fiber of type I_{16}, and C_3, C_4 are contained in other singular fibers. We shall see that the other fixed curve C is smooth too. It suffices to see that both C_3, C_4 are not fixed curves (Then by the same argument as in Case II-(4) we can see that $C = F$ is smooth.) Assume that C_3 is fixed by $f_{r'}$. Then F is reducible and of type I_4. Thus π has singular fibers of type I_{16} and of type I_4, and four sections. By Shioda [14], Corollaries 1.5, 1.7, $det(S_X) = 4$, which is a contradiction. Thus we have the same configuration of fixed curves as in Mukai and Namikawa [10], (4.2). As stated in II-(4), $f_{r'}$ acts trivially on the sublattice of rank 18 generated by 20 nodal curves, and acts as -1 on its orthogonal complement. Hence $f_{r'}$ sends r' to $-r'$.

4.4. *Case* IV-(1). We assume that

$$r' = (2\sum(E_i + F_i + D_i) + \sum G_{ij} - 4\sum D_i)/8.$$

This case is similar to Case II-(1). The involution $f_{r'}$ induced by the involution $(1_{E_{\sqrt{-1}}}, -1_{E_{\sqrt{-1}}})$ acts trivially on the sublattice of rank 18 generated by $\{G_{ij}, E_i, F_j\}$. It follows from Mukai and Namikawa [10], Proposition 2.5, that

$$rank(H^2(X, \mathbb{Z})^{\langle f_{r'} \rangle}) = 18.$$

Recall that r' is orthogonal to G_{ij}, E_i, F_j (see the proof of Lemma 3.6, Case IV-(1)). Hence $f_{r'}$ sends r' to $-r'$.

Case IV-(2). We assume that

$$\begin{aligned} r' = (2\sum(E_i + F_i + D_i) + 4\sum L_j + \sum G_{ij} \\ - 8(L_1 + L_4 + L_6 + L_7 + L_{10} + L_{11}))/16. \end{aligned}$$

This case is similar to Case II-(2). The same elliptic pencil π has three singular fibers of type I_6, one singular fiber of type I_2 and eighteen sections. The same translation induces an automorphism $f_{r'}$ whose action on $\{L_i\}$ is as follows (the action of $f_{r'}$ on $\{G_{ij}, E_i, F_j, D_k\}$ is the same as in Case II-(2)):

$$\begin{aligned}
&f_{r'}(L_5) = L_9, \quad f_{r'}(L_9) = L_1, \quad f_{r'}(L_2) = L_8,\\
&f_{r'}(L_8) = L_4, \quad f_{r'}(L_3) = L_{12}, \quad f_{r'}(L_{12}) = L_6,\\
&f_{r'}(L_1) = G_{11} - L_5 + (G_{12} + G_{21} + E_3 + F_4 + D_3 + D_4 - L_2 - L_3)/2\\
&\qquad + L_8 + L_9 + L_{12},\\
&f_{r'}(L_4) = G_{11} - L_2 + (G_{12} + G_{22} + E_3 + E_4 + F_3 + D_4 - L_3 - L_5)/2\\
&\qquad + L_8 + L_9 + L_{12},\\
&f_{r'}(L_6) = G_{11} - L_3 + (G_{21} + G_{22} + E_4 + F_3 + F_4 + D_3 - L_2 - L_5)/2\\
&\qquad + L_8 + L_9 + L_{12}.
\end{aligned}$$

Note that

$$|\ G_{12} + G_{21} + G_{22} + L_7 + L_{10} + L_{11}\ | = |\ G_{24} + G_{32} + G_{43} + L_2 + L_3 + L_5\ |\ .$$

A direct calculation shows that $f_{r'}(r') = -s'$, where

$$\begin{aligned} s' = (2\sum(E_i + F_i + D_i) + 4\sum L_j + \sum G_{ij} \\ - 8(L_2 + L_3 + L_5 + L_7 + L_{10} + L_{11}))/16. \end{aligned}$$

Case IV-(3). We assume that

$$\begin{aligned} r' = (2\sum G_{ij} &+ \sum (E_i + F_i + D_i) - \sum L_j \\ &- 2(G_{11} + G_{13} + G_{22} + G_{24} + G_{31} + G_{33} + G_{42} + G_{44}) \\ &+ 4(L_1 + L_2 + L_5 + L_7))/8. \end{aligned}$$

This case is similar to Cases II-(4) and III-(3). By the same argument, we have an involution $f_{r'}$ which sends r' to $-r'$.

Case IV-(4). We assume that

$$\begin{aligned} r' = (5\sum G_{ij} &+ 20\sum F_i + 10\sum L_i - 16(G_{22} + G_{24} + G_{41} + G_{43}) \\ &- 24(F_2 + F_4 + L_2 + L_5 + L_8 + L_{11}))/16. \end{aligned}$$

In this case there are exactly twenty-two nodal curves orthogonal to r' whose dual graph is the same graph as that of Case III-(3). Thus we have an involution $f_{r'}$ which sends r' to $-r'$.

5. The automorphism groups

In this section we shall complete a calculation of automorphism groups. First recall that the graph of Leech roots as in Lemma 3.2 is isomorphic to the dual graph Γ of nodal curves on X (see §2). Since w is an interior point of D, w' is also an interior point of a fundamental domain of $W(S_X)^{(2)}$. Hence, modulo $W(S_X)^{(2)}$, we may assume that w' is contained in the Kähler cone $D(X)$, i.e. it is represented by a class of an ample divisor of X. Then each Leech root r as above is represented by an irreducible curve, because $\langle r, w'\rangle = 1$. Hence these Leech roots are represented by nodal curves. Since $Aut(X)$ depends only on the isomorphism class of the lattice S_X (see Lemma 1.1), we can identify these Leech roots with nodal curves as in §2.

Next we remark that the subgroup G_0 of $Aut(D')$ (see Lemma 3.5) can be represented by automorphisms of X. Since G_0 acts on the discriminant A_{S_X} trivially, the action of G_0 on S_X can be extended to the one on $H^2(X, \mathbb{Z})$ acting on T_X trivially. Hence the assertion follows from the Torelli theorem for $K3$ surfaces (Piatetski-Shapiro and Shafarevich [13]).

Case I. We first remark that $O(q_{S_X})$ is isomorphic to $O^+(4, \mathbb{F}_2) \simeq (S_3 \times S_3)\cdot\mathbb{Z}/2\mathbb{Z}$. Let ι be an automorphism of X induced from an automorphism of $E \times F$ given by

$$(x, y) \longrightarrow (x, -y).$$

It is easy to see that the pointwise fixed curves of ι are exactly E_i, F_j. It is known (Mukai and Namikawa [10]) that $\iota^* \mid S_X = 1$ and $\iota^* \mid T_X = -1$. This implies that ι is contained in the center of $Aut(X)$. The quotient surface $X/\langle\iota\rangle$ contains sixteen exceptional curves which are the images of $\{G_{ij}\}$. By contracting these exceptional curves, we have a smooth quadric Q. Thus $Aut(X)/\langle\iota\rangle$ is isomorphic to a subgroup of the group of birational transformations of Q. Denote by $\{t_{ij}\}$ involutions of X induced from the translations of $E \times F$ by 2-torsions (p_i, q_j). Then ι and t_{ij} generate a 2-elementary abelian group of order 32. By Lemma 3.5, we have

Lemma 5.1. *$Aut(D')$ is generated by sixteen involutions t_{ij} and $O^+(4, \mathbb{F}_2)$.*

Let N_1 be a subgroup of $Aut(X)$ generated by twenty-four involutions $f_{r'}$ corresponding to the twenty-four Leech roots in Lemma 3.6 (see 4.1 for $f_{r'}$).

Lemma 5.2. *Let ϕ be an isometry of S_X which preserves the Kähler cone $D(X)$. Then there exists an element $g \in N_1$ such that $g \circ \phi \in Aut(D')$.*

Proof. Since $f_{r'}(r') = -r'$ (see 4.1), $f_{r'}$ sends the positive side with respect to the hyperplane defined by r' to the opposite side. Thus the same proof as in Kondō [8], Lemma 5.3, holds. We remark that this proof works in the remaining Cases II–IV by using the fact that $f_{r'}(r') = -r'$ or $f_{r'}(r') = -s'$, where s' is the projection of a Leech root s of the same type (see §4, Case II-(2), Case III-(1), Case IV-(2)). □

Theorem 5.3. *The automorphism group of $Km(E \times F)$ is a split extension of the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^5$ of order 32 by a group N_1 where $(\mathbb{Z}/2\mathbb{Z})^5$ is generated by translations t_{ij} and ι, and N_1 is generated by twenty-four involutions $f_{r'}$ from Cremona transformation of $\mathbb{P}^2$.*

Proof. By the definition of genericness of X, $Aut(X) \cap Aut(D')$ is a 2-elementary group of order 32 generated by t_{ij} and ι (see Lemmas 1.1, 5.1). On the other hand, for each $f \in Aut(X)$, there exists a $g \in N_1$ with $g \circ f \in Aut(X) \cap Aut(D')$ (Lemma 5.2). Thus we have proved the assersion. □

Case II. In this case, G_0 is generated by $\{t_{ij}\}$ and an automorphism τ of order three (see the proof of Lemma 3.5, Case II). Let σ be an involution of X induced from

$$(x, y) \longrightarrow (y, x)$$

which acts on T_X as -1_{T_X}. Let N_2 be a subgroup of $Aut(X)$ generated by $f_{r'}$ corresponding to the Leech roots as in Lemma 3.6 (also see 4.2 for $f_{r'}$). Then, by the same argument as in Case I, we have

Theorem 5.4. *The automorphism group of $Km(E \times E)$ is a split extension of $((\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathbb{Z}/3\mathbb{Z}) \cdot \mathbb{Z}/2\mathbb{Z}$ by N_2, where the former group is generated by t_{ij}, σ and τ.*

Remark 5.5. ι defined in Case I does not preserve D' and is contained in N_2 (see 4.2, Case II-(1)).

Case III. In this case, $O(q_{S_X}) \simeq S_3 \times \mathbb{Z}/2\mathbb{Z}$ and G_0 is generated by $\{t_{ij}\}$ and automorphisms induced from

$$(x, y) \longrightarrow (y, -x + y), (x, y) \longrightarrow (\omega(-x + y), -\omega^2 y).$$

We shall find an automorphism of order 6 which preserves D' and whose image in $O(q_{S_X})$ has order 6 (see Lemma 1.1). Let σ be an involution of S_X defined by

$$E_1 \leftrightarrow G_{24}, \quad E_2 \leftrightarrow G_{43}, \quad E_3 \leftrightarrow G_{32}, \quad E_4 \leftrightarrow G_{11},$$

$$F_1 \leftrightarrow G_{44}, \quad F_2 \leftrightarrow G_{31}, \quad F_3 \leftrightarrow G_{23}, \quad F_4 \leftrightarrow G_{12},$$

$$D_1 \leftrightarrow G_{14}, \quad D_2 \leftrightarrow G_{42}, \quad D_3 \leftrightarrow G_{21}, \quad D_4 \leftrightarrow G_{33},$$

$$C_1 \leftrightarrow G_{34}, \quad C_2 \leftrightarrow G_{22}, \quad C_3 \leftrightarrow G_{41}, \quad C_4 \leftrightarrow G_{13}.$$

Note that σ interchanges two families $\{G_{ij}\}$ and $\{E_i, F_j, D_k, C_l\}$, and σ is in fact an isometry of S_X because these curves generate S_X (Lemma 2.1). Recall that

$A_{S_X} = S_X^*/S_X \simeq (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/3\mathbb{Z}$. Since σ is an involution, it acts on the 2-Sylow subgroup of A_{S_X} trivially. On the other hand we can see that the class

$$\delta = \sum (G_{ij} + 2(E_k + F_l + D_m + C_n))/3$$

is a generator of the 3-Sylow subgroup of A_{S_X}, and $\sigma(\delta) = -\delta$. Thus σ can be extended to an isometry of $H^2(X,\mathbb{Z})$ with $\sigma \mid T_X = -1_{T_X}$. Since σ preserves the projection of the Weyl vector (Lemma 3.4), it preserves the Kähler cone too. It now follows from the Torelli theorem that σ is represented by an automorphism of order two. Let τ be an automorphism of order 3 induced from

$$(x, y) \longrightarrow (\omega x, \omega y).$$

Then $\sigma \circ \tau$ is the desired automorphism.

Let N_3 be a subgroup of $Aut(X)$ generated $f_{r'}$ corresponding to the Leech roots as in Lemma 3.6. Then we have

Theorem 5.6. *The group of automorphisms of $Km(E_\omega \times E_\omega)$ is a split extension of $((\mathbb{Z}/2\mathbb{Z})^4 \cdot A_4) \cdot \mathbb{Z}/6\mathbb{Z}$ by N_3.*

Case IV. In this case, $O(q_{S_X}) \simeq D_8$ and G_0 is generated by translations by sections of five elliptic pencils with six singular fibers of type I_4. On the other hand, as we mentioned in the proof of Lemma 3.5, there is an isometry in $Aut(D)$ acting on $R = A_3 \oplus A_3$ as automorphism of order 4. This induces an isometry ϕ of S_X preserveing D'. Then ϕ can be extended an isometry of $H^2(X,\mathbb{Z})$ acting on T_X as an isometry of order 4. It now follows from the Torelli theorem that ϕ is represented by an automorphism of order 4.

Let N_4 be a subgroup of $Aut(X)$ generated by $f_{r'}$ corresponding to the Leech roots as in Lemma 3.6. Then we have

Theorem 5.7. *The group of automorphisms of $Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$ is a split extension of $((\mathbb{Z}/2\mathbb{Z})^4 \cdot A_5) \cdot \mathbb{Z}/4\mathbb{Z}$ by N_4.*

Remark 5.8. Let M_{24} be the Mathieu group, which acts naturally on a set Ω of twenty-four letters. Let M_{20} be the pointwise stabilizer group of four points in Ω. Then M_{20} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \cdot A_5$. In [9] the second author has proved that $((\mathbb{Z}/2\mathbb{Z})^4 \cdot A_5) \cdot \mathbb{Z}/4\mathbb{Z}$ has the maximum order among all finite groups of automorphisms of $K3$ surfaces.

Remark 5.9. The method in this paper is applicable to other $K3$ surfaces, for example, the Fermat quartic surface. In this case there exists a primitive embedding of the Picard lattice into $II_{1,25}$ such that forty-eight lines on the Fermat quartic surface appear as a part of faces of the corresponding D'. The remaining problem is to find an automorphism corresponding to each of the remaining faces of D' like $f_{r'}$ as above.

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